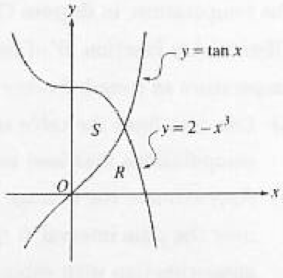
AP® CALCULUS AB
2001 SCORING GUIDELINES

Question 1

Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.



(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the x-axis.

Point of intersection
$2 - x^3 = \tan x$ at $(A, B) = (0.902155, 1.265751)$

(a) Area $R = \int_0^A \tan x\,dx + \int_A^{\sqrt[3]{2}} (2 - x^3)\,dx = 0.729$

or

Area $R = \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy = 0.729$

or

Area $R = \int_0^{\sqrt[3]{2}} (2 - x^3)\,dx - \int_0^A (2 - x^3 - \tan x)\,dx = 0.729$

3 : { 1 : limits; 1 : integrand; 1 : answer }

(b) Area $S = \int_0^A (2 - x^3 - \tan x)\,dx = 1.160$ or 1.161

or

Area $S = \int_0^B \tan^{-1} y\,dy + \int_B^2 (2-y)^{1/3}\,dy = 1.160$ or 1.161

or

Area S

$= \int_0^2 (2-y)^{1/3}\,dy - \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy$

$= 1.160$ or 1.161

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) Volume $= \pi \int_0^A \left((2 - x^3)^2 - \tan^2 x\right) dx$

$= 2.652\pi$ or 8.331 or 8.332

3 : { 1 : limits and constant; 1 : integrand; 1 : answer }

Question 2

The temperature, in degrees Celsius (°C), of the water in a pond is a differentiable function W of time t. The table above shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function P, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t, where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \le t \le 15$ days.

(a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \text{ °C/day or}$$

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \text{ °C/day}$$

2 : { 1 : difference quotient; 1 : answer (with units) }

(b) $\frac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

Average temperature $\approx \frac{1}{15}(376.5) = 25.1$ °C

2 : { 1 : trapezoidal method; 1 : answer }

(c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}\Big|_{t=12}$

$= -30e^{-4} = -0.549$ °C/day

This means that the temperature is decreasing at the rate of 0.549 °C/day when $t = 12$ days.

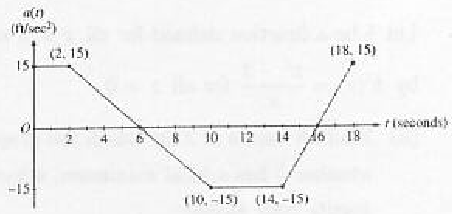
2 : { 1 : $P'(12)$ (with or without units); 1 : interpretation }

(d) $\frac{1}{15}\int_0^{15}\left(20 + 10te^{-t/3}\right)dt = 25.757$ °C

3 : { 1 : integrand; 1 : limits and average value constant; 1 : answer }

Question 3

A car is traveling on a straight road with velocity 55 ft/sec at time $t = 0$. For $0 \le t \le 18$ seconds, the car's acceleration $a(t)$, in ft/sec², is the piecewise linear function defined by the graph above.



(a) Is the velocity of the car increasing at $t = 2$ seconds? Why or why not?

(b) At what time in the interval $0 \le t \le 18$, other than $t = 0$, is the velocity of the car 55 ft/sec? Why?

(c) On the time interval $0 \le t \le 18$, what is the car's absolute maximum velocity, in ft/sec, and at what time does it occur? Justify your answer.

(d) At what times in the interval $0 \le t \le 18$, if any, is the car's velocity equal to zero? Justify your answer.

(a) Since $v'(2) = a(2)$ and $a(2) = 15 > 0$, the velocity is increasing at $t = 2$.	1 : answer and reason
(b) At time $t = 12$ because $v(12) - v(0) = \int_0^{12} a(t)\,dt = 0$.	2 : { 1 : $t = 12$; 1 : reason }
(c) The absolute maximum velocity is 115 ft/sec at $t = 6$. The absolute maximum must occur at $t = 6$ or at an endpoint. $v(6) = 55 + \int_0^6 a(t)\,dt$ $= 55 + 2(15) + \frac{1}{2}(4)(15) = 115 > v(0)$ $\int_6^{18} a(t)\,dt < 0$ so $v(18) < v(6)$	4 : { 1 : $t = 6$; 1 : absolute maximum velocity; 1 : identifies $t = 6$ and $t = 18$ as candidates or indicates that v increases, decreases, then increases; 1 : eliminates $t = 18$ }
(d) The car's velocity is never equal to 0. The absolute minimum occurs at $t = 16$ where $v(16) = 115 + \int_6^{16} a(t)\,dt = 115 - 105 = 10 > 0$.	2 : { 1 : answer; 1 : reason }

Question 4

Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$

$h'(x)$	$-$	0	$+$	und	$-$	0	$+$
x		$-\sqrt{2}$		0		$\sqrt{2}$	

Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

$4: \begin{cases} 1: x = \pm\sqrt{2} \\ 1: \text{analysis} \\ 2: \text{conclusions} \\ \quad <-1> \text{ not dealing with discontinuity at } 0 \end{cases}$

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of h is concave up for all $x \neq 0$.

$3: \begin{cases} 1: h''(x) \\ 1: h''(x) > 0 \\ 1: \text{answer} \end{cases}$

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of h is concave up for $x > 4$.

1 : answer with reason

Question 5

A cubic polynomial function f is defined by

$$f(x) = 4x^3 + ax^2 + bx + k$$

where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$.

(a) Find the values of a and b.

(b) If $\int_0^1 f(x)\,dx = 32$, what is the value of k ?

(a) $f'(x) = 12x^2 + 2ax + b$ $f''(x) = 24x + 2a$ $f'(-1) = 12 - 2a + b = 0$ $f''(-2) = -48 + 2a = 0$ $a = 24$ $b = -12 + 2a = 36$	$5: \begin{cases} 1: f'(x) \\ 1: f''(x) \\ 1: f'(-1) = 0 \\ 1: f''(-2) = 0 \\ 1: a,\ b \end{cases}$
(b) $\int_0^1 \left(4x^3 + 24x^2 + 36x + k\right) dx$ $= x^4 + 8x^3 + 18x^2 + kx\Big\|_{x=0}^{x=1} = 27 + k$ $27 + k = 32$ $k = 5$	$4: \begin{cases} 2: \text{antidifferentiation} \\ \quad < -1 > \text{each error} \\ 1: \text{expression in } k \\ 1: k \end{cases}$

Question 6

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

(a) $$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6 - 2x) - 2y^2$$
$$= 2y^3(6 - 2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3, \frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

3 : { 2 : $\frac{d^2y}{dx^2}$
< −2 > product rule or chain rule error
1 : value at $\left(3, \frac{1}{4}\right)$ }

(b) $$\frac{1}{y^2}\,dy = (6 - 2x)\,dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$

$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

6 : { 1 : separates variables
1 : antiderivative of dy term
1 : antiderivative of dx term
1 : constant of integration
1 : uses initial condition $f(3) = \frac{1}{4}$
1 : solves for y }

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables