Question 1

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

(a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

(c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

(a) Area $= \int_{1/2}^{1} (e^x - \ln x)\,dx = 1.222$ or 1.223

$2 \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) Volume $= \pi \int_{1/2}^{1} \left((4 - \ln x)^2 - (4 - e^x)^2 \right) dx$

$= 7.515\pi$ or 23.609

$4 \begin{cases} 1: \text{limits and constant} \\ 2: \text{integrand} \\ \quad < -1 > \text{each error} \\ \quad \text{Note: } 0/2 \text{ if not of the form} \\ \quad k\int_a^b \left(R(x)^2 - r(x)^2 \right) dx \\ 1: \text{answer} \end{cases}$

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$

$x = 0.567143$

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$
$h(0.5) = 2.3418$
$h(1) = 2.718$

The absolute minimum is 2.330.
The absolute maximum is 2.718.

$3 \begin{cases} 1: \text{considers } h'(x) = 0 \\ 1: \text{identifies critical point} \\ \quad \text{and endpoints as candidates} \\ 1: \text{answers} \end{cases}$

Note: Errors in computation come off the third point.

Question 2

The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{\left(t^2 - 24t + 160\right)}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{\left(t^2 - 38t + 370\right)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open. At time $t = 9$, there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round answer to the nearest whole number.

(b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_9^t (E(x) - L(x))\,dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?

Solution	Scoring
(a) $\int_9^{17} E(t)\,dt = 6004.270$ 6004 people entered the park by 5 pm.	3: { 1: limits 1: integrand 1: answer }
(b) $15\int_9^{17} E(t)\,dt + 11\int_{17}^{23} E(t)\,dt = 104048.165$ The amount collected was \$104,048. or $\int_{17}^{23} E(t)\,dt = 1271.283$ 1271 people entered the park between 5 pm and 11 pm, so the amount collected was $\$15 \cdot (6004) + \$11 \cdot (1271) = \$104,041.$	1: setup
(c) $H'(17) = E(17) - L(17) = -380.281$ There were 3725 people in the park at $t = 17$. The number of people in the park was decreasing at the rate of approximately 380 people/hr at time $t = 17$.	3: { 1: value of $H'(17)$ 2: meanings { 1: meaning of $H(17)$ 1: meaning of $H'(17)$ $< -1 >$ if no reference to $t = 17$ } }
(d) $H'(t) = E(t) - L(t) = 0$ $t = 15.794$ or 15.795	2: { 1: $E(t) - L(t) = 0$ 1: answer }

Question 3

An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(a) What is the acceleration of the object at time $t = 4$?

(b) Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.

Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

(c) What is the total distance traveled by the object over the time interval $0 \leq t \leq 4$?

(d) What is the position of the object at time $t = 4$?

Solution	Scoring
(a) $a(4) = v'(4) = \frac{\pi}{3}\cos\left(\frac{4\pi}{3}\right)$ $= -\frac{\pi}{6}$ or -0.523 or -0.524	1 : answer
(b) On $3 < t < 4.5$: $a(t) = v'(t) = \frac{\pi}{3}\cos\left(\frac{\pi}{3}t\right) < 0$ Statement I is correct since $a(t) < 0$. Statement II is correct since $v(t) < 0$ and $a(t) < 0$.	3 { 1 : I correct, with reason 1 : II correct 1 : reason for II
(c) Distance $= \int_0^4 \lvert v(t)\rvert\, dt = 2.387$ OR $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$ $x(0) = 2$ $x(4) = 2 + \frac{9}{2\pi} = 3.43239$ $v(t) = 0$ when $t = 3$ $x(3) = \frac{6}{\pi} + 2 = 3.90986$ $\lvert x(3) - x(0)\rvert + \lvert x(4) - x(3)\rvert = \frac{15}{2\pi} = 2.387$	3 { 1 : { limits of 0 and 4 on an integral of $v(t)$ or $\lvert v(t)\rvert$ or uses $x(0)$ and $x(4)$ to compute distance } 1 : handles change of direction at student's turning point 1 : answer 0/1 if incorrect turning point or no turning point
(d) $x(4) = x(0) + \int_0^4 v(t)\, dt = 3.432$	2 { 1 : integral 1 : answer
OR $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$ $x(4) = 2 + \frac{9}{2\pi} = 3.432$	OR 2 { 1 : $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + C$ 1 : answer 0/1 if no constant of integration

Question 4

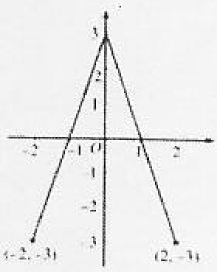
The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.

(c) For what values of x in the open interval $(-2, 2)$ is the graph of g concave down? Explain your reasoning.

(d) On the axes provided, sketch the graph of g on the closed interval $[-2, 2]$.



Graph of f

(a) $g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt = -\frac{3}{2}$

$g'(-1) = f(-1) = 0$

$g''(-1) = f'(-1) = 3$

3 { 1 : $g(-1)$; 1 : $g'(-1)$; 1 : $g''(-1)$ }

(b) g is increasing on $-1 < x < 1$ because $g'(x) = f(x) > 0$ on this interval.

2 { 1 : interval; 1 : reason }

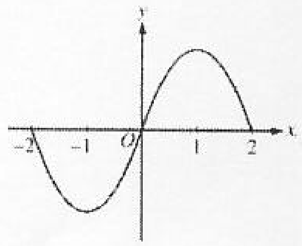
(c) The graph of g is concave down on $0 < x < 2$ because $g''(x) = f'(x) < 0$ on this interval.

or

because $g'(x) = f(x)$ is decreasing on this interval.

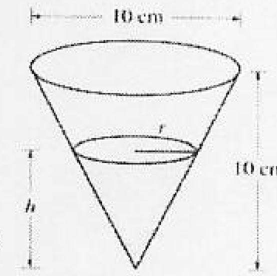
2 { 1 : interval; 1 : reason }

(d)



2 { 1 : $g(-2) = g(0) = g(2) = 0$; 1 : appropriate increasing/decreasing and concavity behavior; $< -1 >$ vertical asymptote }

Question 5



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

(a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi \text{ cm}^3$

1 : V when $h = 5$

(b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3;\ \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi \text{ cm}^3/\text{hr}$$

OR

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right);\ \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5,\,r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left(\frac{25}{4}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$$

$$= -\frac{15}{8}\pi \text{ cm}^3/\text{hr}$$

5:
- 1 : $r = \frac{1}{2}h$ in (a) or (b)
- 1 : V as a function of one variable in (a) or (b) OR $\frac{dr}{dt}$
- 2 : $\frac{dV}{dt}$
  < −2 > chain rule or product rule error
- 1 : evaluation at $h = 5$

(c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} = -\frac{3}{40}\pi h^2$

$$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10}\cdot\text{area}$$

The constant of proportionality is $-\frac{3}{10}$.

2:
- 1 : shows $\frac{dV}{dt} = k\cdot\text{area}$
- 1 : identifies constant of proportionality

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units of cm^3 in (a) and cm^3/hr in (b)

1 : correct units in (a) and (b)

Question 6

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \le x \le 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(a) Evaluate $\int_0^{1.5} (3f'(x) + 4)\,dx$. Show the work that leads to your answer.

(b) Write an equation of the line tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

(c) Find a positive real number r having the property that there must exist a value c with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.

(d) Let g be the function given by $g(x) = \begin{cases} 2x^2 - x - 7 & \text{for } x < 0 \\ 2x^2 + x - 7 & \text{for } x \ge 0. \end{cases}$

The graph of g passes through each of the points $(x, f(x))$ given in the table above. Is it possible that f and g are the same function? Give a reason for your answer.

(a) $\int_0^{1.5} (3f'(x) + 4)\,dx = 3\int_0^{1.5} f'(x)\,dx + \int_0^{1.5} 4\,dx$

$= 3f(x) + 4x\Big|_0^{1.5} = 3(-1 - (-7)) + 4(1.5) = 24$

2: { 1 : antiderivative; 1 : answer }

(b) $y = 5(x - 1) - 4$

$f(1.2) \approx 5(0.2) - 4 = -3$

The approximation is less than $f(1.2)$ because the graph of f is concave up on the interval $1 < x < 1.2$.

3: { 1 : tangent line; 1 : computes y on tangent line at $x = 1.2$; 1 : answer with reason }

(c) By the Mean Value Theorem there is a c with $0 < c < 0.5$ such that

$f''(c) = \dfrac{f'(0.5) - f'(0)}{0.5 - 0} = \dfrac{3 - 0}{0.5} = 6 = r$

2: { 1 : reference to MVT for f' (or differentiability of f'); 1 : value of r for interval $0 \le x \le 0.5$ }

(d) $\lim\limits_{x \to 0^-} g'(x) = \lim\limits_{x \to 0^-} (4x - 1) = -1$

$\lim\limits_{x \to 0^+} g'(x) = \lim\limits_{x \to 0^+} (4x + 1) = +1$

Thus g' is not continuous at $x = 0$, but f' is continuous at $x = 0$, so $f \ne g$.

OR

$g''(x) = 4$ for all $x \ne 0$, but it was shown in part (c) that $f''(c) = 6$ for some $c \ne 0$, so $f \ne g$.

2: { 1 : answers "no" with reference to g' or g''; 1 : correct reason }